

Infinite square-free self-shuffling words

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Abstract

In this paper we answer two recent questions from [2] and [5] about self-shuffling words. An infinite word w is called self-shuffling, if $w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i$ for some finite words U_i, V_i . Harju [5] recently asked whether square-free self-shuffling words exist. We answer this question affirmatively. Besides that, we build an infinite word such that no word in its shift orbit closure is self-shuffling, answering positively a question from [2].

Keywords: infinite words, shuffling, square-free words, shift orbit closure, self-shuffling words

1 Introduction

A self-shuffling word, a notion which was recently introduced by Charlier et al. [2], is an infinite word that can be reproduced by shuffling it with itself. More formally, given infinite words $x, y \in \Sigma^{\omega}$ over a finite alphabet Σ , we define $\mathcal{S}(x, y) \subseteq \Sigma^{\omega}$ to be the collection of all infinite words z for which there exists a factorization

$$z = \prod_{i=0}^{\infty} U_i V_i$$

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with each $U_i, V_i \in \Sigma^*$ and with $x = \prod_{i=0}^{\infty} U_i$, $y = \prod_{i=0}^{\infty} V_i$. An infinite word $w \in \Sigma^{\omega}$ is *self-shuffling* if $w \in \mathcal{S}(w, w)$. Various well-known words, e.g., the Thue-Morse word or the Fibonacci word, were shown to be self-shuffling.

Harju [5] studied shuffles of both finite and infinite square-free words, i.e., words that have no factor of the form uu for some non-empty factor u . More results on square-free shuffles were obtained independently by Harju and Müller [6], and Currie and Saari [4]. However, the question about the existence of an infinite square-free self-shuffling word, posed in [5], remained open. We give a positive answer to this question in Sections 2 and 3.

The *shift orbit closure* S_w of an infinite word w can be defined, e.g., as the set of infinite words whose sets of factors are contained in the set of factors of w . In [2] it has been proved that each word has a non-self-shuffling word in its shift orbit closure, and the following question has been asked: Does there exist a word for which no element of its shift orbit closure is self-shuffling (Question 7.2)? In Section 4 we provide a positive answer to the question. More generally, we show the existence of a word such that for any three words x, y, z in its shift orbit closure, if x is a shuffle of y and z , then the three words are pairwise different. On the other hand, we show that for any infinite word there exist three different words x, y, z in its shift orbit closure such that $x \in \mathcal{S}(y, z)$ (see Proposition 7).

Apart from the usual concepts in combinatorics on words, which can be found for instance in the book of Lothaire [7], we make use of the following notations: For every $k \geq 1$, we denote the alphabet $\{0, 1, \dots, k-1\}$ by Σ_k . For a word $w = uvz$ we say that u is a *prefix* of w , v is a *factor* of w , and z is a *suffix* of w . We denote these prefix- and suffix relations by $u \leq_p w$ and $v \leq_s w$, respectively. By $w[i, j]$ we denote the factor of w starting at position i and ending after position j . Note that we start numbering the positions with 0.

A *prefix code* is a set of words with the property that none of its elements is a prefix of another element. Similarly, a *suffix code* is a set of words where no element is a suffix of another one. A *bifix code* is a set that is both a prefix code and a suffix code. A morphism h is *square-free* if for all square-free words w , the image $h(w)$ is square-free.

2 A square-free self-shuffling word on four letters

Let $g : \Sigma_4^* \rightarrow \Sigma_4^*$ be the morphism defined as follows:

$$\begin{aligned} g(0) &= 0121, \\ g(1) &= 032, \\ g(2) &= 013, \\ g(3) &= 0302. \end{aligned}$$

We will show that the fixed point $w = g^{\omega}(0)$ is square-free and self-shuffling in the following. Note that g is not a square-free morphism, that is, it does not preserve square-freeness, as $g(23) = 0130302$ contains the square 3030.

Lemma 1. *The word $w = g^\omega(0)$ contains no factor of the form $3u1u3$ for some $u \in \Sigma_4^*$.*

Proof. We assume that there exists a factor of the form $3u1u3$ in w , for some word $u \in \Sigma_4^*$. From the definition of g , we observe that u can not be empty. Furthermore, we see that every 3 in w is preceded by either 0 or 1. If $1 \leq_s u$, then we had an occurrence of the factor 11 in w , which is not possible by the definition of g , hence $0 \leq_s u$. Now, every 3 is followed by either 0 or 2 in w and 01 is followed by either 2 or 3. Since both $3u$ and $01u$ are factors of w , we must have $2 \leq_p u$. This means that the factor 012 appears at the center of $u1u$, which can only be followed by 1 in w , thus $21 \leq_p u$. However, this results in the factor 321 as a prefix of $3u1u3$, which does not appear in w , as seen from the definition of g . \square

Lemma 2. *The word $w = g^\omega(0)$ is square-free.*

Proof. We first observe that $\{g(0), g(1), g(2), g(3)\}$ is a bifix code. Furthermore, we can verify that there are no squares uu with $|u| \leq 3$ in w . Let us assume now, that the square uu appears in w and that u is the shortest word with this property. If $u = 02u'$, then $u' = u''03$ must hold, since 02 appears only as a factor of $g(3)$, and thus uu is a suffix of the factor $g(3)u''g(3)u''$ in w . As $w = g(w)$, also the shorter square $3g^{-1}(u'')3g^{-1}(u'')$ appears in w , a contradiction. The same desubstitution principle also leads to occurrences of shorter squares in w if $u = xu'$ and $x \in \{01, 03, 10, 12, 13, 21, 30, 32\}$.

If $u = 2u'$ then either $03 \leq_s u$ or $030 \leq_s u$ or $01 \leq_s u$, by the definition of g . In the last case, that is when $01 \leq_s u$, we must have $21 \leq_p u$, which is covered by the previous paragraph. If $u' = u''030$, then uu is followed by 2 in w and we can desubstitute to obtain the shorter square $g^{-1}(u'')3g^{-1}(u'')3$ in w . If $u = 2u'$ and $u' = u''03$, and uu is preceded by 03 or followed by 2 in w , we can desubstitute to $1g^{-1}(u'')1g^{-1}(u'')$ or $g^{-1}(u'')1g^{-1}(u'')1$, respectively. Therefore, assume that $u = 2u''03$ and uu is preceded by 030 and followed by 02 in w . This however means that we can desubstitute to get an occurrence of the factor $3g^{-1}(u'')1g^{-1}(u'')3$ in w , a contradiction to Lemma 1. \square

We now show that $w = g^\omega(0)$ can be written as $w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i$ with $U_i, V_i \in \Sigma_4^*$.

Lemma 3. *The word $w = g^\omega(0)$ is self-shuffling.*

Proof. In what follows we use the notation $x = v^{-1}u$ meaning that $u = vx$ for finite words x, u, v . We are going to show that the self-shuffle is given by the following:

$$\begin{array}{llll}
U_0 = g^2(0), & U_1 = 0, & \dots, & U_{6i+2} = g^i(0^{-1}g(0)0), & U_{6i+3} = g^i(0^{-1}g(3)0), \\
& & & U_{6i+4} = g^i(0^{-1}g(201)0), & U_{6i+5} = g^i(30), \\
& & & U_{6i+6} = g^i(2g(03)), & U_{6i+7} = g^{i+1}(20), \\
V_0 = g(0)03, & V_1 = 2g(2)0, & \dots, & V_{6i+2} = g^i(0^{-1}g(1)0), & V_{6i+3} = g^i(0^{-1}g(03)0), \\
& & & V_{6i+4} = g^i(1), & V_{6i+5} = g^i(3), \\
& & & V_{6i+6} = g^{i+1}(0), & V_{6i+7} = g^{i+1}(0^{-1}g(2)0).
\end{array}$$

Now we verify that

$$w = \prod_{i=0}^{\infty} U_i V_i = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i,$$

from which it follows that w is self-shuffling. It suffices to show that each of the above products is fixed by g . Indeed, straightforward computations show that

$$\prod_{i=0}^{\infty} U_i = g^2(0)g^2(121)g^3(121)\cdots,$$

which is fixed by g :

$$\begin{aligned} g\left(\prod_{i=0}^{\infty} U_i\right) &= g\left(g^2(0)g^2(121)g^3(121)\cdots\right) = g^3(0)g^3(121)g^4(121)\cdots \\ &= g^2(0121)g^3(121)g^4(121)\cdots = g^2(0)g^2(121)g^3(121)\cdots = \prod_{i=0}^{\infty} U_i, \end{aligned}$$

hence $\prod_{i=0}^{\infty} U_i$ is fixed by g and thus $w = \prod_{i=0}^{\infty} U_i$. In a similar way we show that $w = \prod_{i=0}^{\infty} V_i = \prod_{i=0}^{\infty} U_i V_i$. \square

3 Square-free self-shuffling words on three letters

We remark that we can immediately produce a square-free self-shuffling word over Σ_3 from $g^\omega(0)$: Charlier et al. [2] noticed that the property of being self-shuffling is preserved by the application of a morphism. Furthermore, Brandenburg [1] showed that the morphism $f : \Sigma_4^* \rightarrow \Sigma_3^*$, defined by

$$\begin{aligned} f(0) &= 010201202101210212, \\ f(1) &= 010201202102010212, \\ f(2) &= 010201202120121012, \\ f(3) &= 010201210201021012, \end{aligned}$$

is square-free. Therefore, the word $f(g^\omega(0))$ is a ternary square-free self-shuffling word, from which we can produce a multitude of others by applying square-free morphisms from Σ_3^* to Σ_3^* .

4 A word with non self-shuffling shift orbit closure

In this section we provide a positive answer to the question from [2] whether there exists a word for which no element of its shift orbit closure is self-shuffling.

The *Hall word* $\mathcal{H} = 012021012102\cdots$ is defined as the fixed point of the morphism $h(0) = 012, h(1) = 02, h(2) = 1$. Sometimes it is referred to as a *ternary Thue-Morse*

word. It is well known that this word is square-free. We show that no word in the shift orbit closure $S_{\mathcal{H}}$ of the Hall word is self-shuffling. More generally, we show that if x is a shuffle of y and z for $x, y, z \in S_{\mathcal{H}}$, then they are pairwise different.

Proposition 4. *There are no words x, y in the shift orbit closure of the Hall word such that $x \in \mathcal{S}(y, y)$.*

Proof. Suppose the converse, i.e., there exist words $x, y \in S_{\mathcal{H}}$ such that

$$x = \prod_{i=0}^{\infty} U_i V_i, \quad y = \prod_{i=0}^{\infty} U_i = \prod_{i=0}^{\infty} V_i.$$

Define the set X of infinite words as follows:

$$X = \{x \in S_{\mathcal{H}} \mid x \in \mathcal{S}(y, y) \text{ for some } y \in S_{\mathcal{H}}\}.$$

In other words, X consists of words in $S_{\mathcal{H}}$ which can be introduced as a shuffle of some word y in $S_{\mathcal{H}}$ with itself. Now suppose, for the sake of contradiction, that X is non empty, and consider $x \in X$ with the first block U_0 of the smallest possible positive length. We remark that such x and corresponding y are not necessarily unique.

We can suppose without loss of generality that y starts with 0 or 10. Otherwise, we exchange 0 and 2, consider the morphism $0 \mapsto 1, 1 \mapsto 20, 2 \mapsto 210$, and the argument is symmetric.

It is not hard to see from the properties of the morphism h that removing every occurrence of 1 from y results in $(02)^\omega$. Hence the blocks in the factorizations of y after removal of 1 are of the form $(02)^i$ for some integer i . Thus the first letter of each block U_i and V_i that is different from 1 is 0, and the last letter different from 1 is 2.

Then, U_i and V_i are images by the morphism h of factors of the fixed point of h . Therefore, there are words $x', y' \in S_{\mathcal{H}}$ such that $x = h(x'), y = h(y'), U_i = h(U'_i), V_i = h(V'_i)$, and $x' = \prod_{i=0}^{\infty} U'_i V'_i, y' = \prod_{i=0}^{\infty} U'_i = \prod_{i=0}^{\infty} V'_i$.

Notice that the first block U_0 cannot be equal to 1. Indeed, otherwise x starts with 11, which is impossible, since 11 is not a factor of the fixed point of h .

Clearly, taking the preimage decreases the lengths of blocks in the factorization (except for those equal to 1), and since $U_0 \neq 1$, the length of the first block in the preimage is smaller, i.e., $|U'_0| < |U_0|$. This is a contradiction with the minimality of $|U_0|$. \square

Corollary 5. *There are no self-shuffling words in the shift orbit closure of \mathcal{H} .*

With a similar argument we can prove the following:

Proposition 6. *There are no words x, y in the shift orbit closure of \mathcal{H} such that $x \in \mathcal{S}(x, y)$.*

Proof. First we introduce a notation $x \in \mathcal{S}_2(y, z)$, meaning that there exists a shuffle starting with the word z (i.e., $U_0 = \varepsilon, V_0 \neq \varepsilon$). Next, $x \in \mathcal{S}(x, y)$ implies that there exists z in the same shift orbit closure such that $z \in \mathcal{S}_2(z, y)$. Indeed, one can remove

the prefix U_0 of x to get z , i.e., $z = (U_0)^{-1}x$, and keep all the other blocks U_i, V_i in the shuffle product.

Define the set Z of infinite words as follows:

$$Z = \{z \in S_{\mathcal{H}} \mid z \in \mathcal{S}_2(z, y) \text{ for some } y \in S_{\mathcal{H}}\}.$$

In other words, Z consists of words in $S_{\mathcal{H}}$ which can be introduced as a shuffle of some word y in $S_{\mathcal{H}}$ with z starting with the block V_0 . Now consider $z \in Z$ with the first block V_0 of the smallest possible length. We remark that such z and a corresponding y are not necessarily unique.

As in the proof of Proposition 4, the shuffle cannot start with a block of length 1. Again, if we remove every occurrence of 1 in y (and in z), we get $(02)^\omega$ or $(20)^\omega$; moreover, since V_0 contains letters different from 1, the first letter different from 1 is the same in y and z . So, without loss of generality we assume that both y and z without 1 are $(02)^\omega$, and the blocks U_i and V_i without 1 are integer powers of 02. Then, U_i and V_i are images by the morphism h of factors of \mathcal{H} . Therefore, there are words $z', y' \in S_{\mathcal{H}}$ such that $z = h(z'), y = h(y'), U_i = h(U'_i), V_i = h(V'_i)$, and $z' = \prod_{i=0}^{\infty} (U'_i V'_i) = \prod_{i=0}^{\infty} V'_i$, $y' = \prod_{i=0}^{\infty} U'_i$ (i.e., $z' \in Z$).

As in the proof of Proposition 4, since $V_0 \neq 1$, the length of the first block in the preimage is smaller, i.e., $|V'_0| < |V_0|$. This is again a contradiction with the minimality of $|V_0|$. \square

So, we proved that if there are three words x, y, z in the shift orbit closure of the fixed point of h such that $x \in \mathcal{S}(y, z)$, then they should be pairwise distinct. Now we are going to prove that for any infinite word there exist three different words in its shift orbit closure such that $x \in \mathcal{S}(y, z)$.

An infinite word x is called *recurrent*, if each its prefix occurs infinitely many times in it.

Proposition 7. *Let x be a recurrent infinite word. Then there exist two words y, z in the shift orbit closure of x such that $x \in \mathcal{S}(y, z)$.*

Proof. We build the shuffle inductively.

Start from any prefix U_0 of x . Since x is recurrent, each of its prefixes occurs infinitely many times in it. Find another occurrence of U_0 in x and denote its position by i_1 . Put $V_0 = x[|U_0|, i_1 + |U_0| - 1]$.

At step k , suppose that the shuffle of the prefix of x is built:

$$x[0, \Sigma_{l=0}^{k-1} (|U_l| + |V_l|) - 1] = \prod_{l=0}^{k-1} U_l V_l, \quad y[0, \Sigma_{l=0}^{k-1} |U_l| - 1] = \prod_{l=0}^{k-1} U_l, \quad z[0, \Sigma_{l=0}^{k-1} |V_l| - 1] = \prod_{l=0}^{k-1} V_l,$$

such that $\prod_{l=0}^{k-1} U_l$ is the suffix of $x[0, \Sigma_{l=0}^{k-1} (|U_l| + |V_l|) - 1] = \prod_{l=0}^{k-1} U_l V_l$ starting at position i_{k-1} , and $\prod_{l=0}^{k-1} V_l$ is the suffix of $x[0, \Sigma_{l=0}^{k-1} (|U_l| + |V_l|) - 1] = \prod_{l=0}^{k-1} U_l V_l$ starting at position j_{k-1} .

Find another occurrence of $\prod_{l=0}^{k-1} V_l$ in x at some position $j_k > j_{k-1}$. We can do it since x is recurrent. Put $U_k = x[\sum_{l=0}^{k-1} (|U_l| + |V_l|), j_k - 1 + \sum_{l=0}^{k-1} |V_l|]$. We note that $\prod_{l=0}^k U_l$ is a factor of x by the construction; more precisely, it occurs at position i_{k-1} .

Find an occurrence of $\prod_{l=0}^k U_l$ at some position $i_k > i_{k-1}$, put $V_k = x[\sum_{l=0}^{k-1} (|U_l| + |V_l|) + |U_k|, i_k - 1 + \sum_{l=0}^k |U_l|]$. As above, $\prod_{l=0}^k V_l$ is a factor of x by the construction since it occurs at position j_{k-1} . Moreover, both $\prod_{l=0}^k U_l$ and $\prod_{l=0}^k V_l$ are suffixes of $x[0, \sum_{l=0}^k (|U_l| + |V_l|) - 1] = \prod_{i=0}^k U_i V_i$.

Continuing this line of reasoning, we build the required factorization. \square

Since each infinite word contains a recurrent (actually, even a uniformly recurrent) word in its shift orbit closure, we obtain the following corollary:

Corollary 8. *Each infinite word w contains words x, y, z in its shift orbit closure such that $x \in \mathcal{S}(y, z)$.*

The following example shows that the recurrence condition in Proposition 7 cannot be omitted:

Example 9. Consider the word $3\mathcal{H} = 3012021 \dots$ which is obtained from \mathcal{H} by adding a letter 3 in the beginning. Then the shift orbit closure of $3\mathcal{H}$ consists of the shift orbit closure of \mathcal{H} and the word $3\mathcal{H}$ itself. Assuming $3\mathcal{H}$ is a shuffle of two words in its shift orbit closure, one of them is $3\mathcal{H}$ (there are no other 3's) and the other one is something in the shift orbit closure of \mathcal{H} , we let y denote this other word. Clearly, the shuffle starts with $3\mathcal{H}$, and cutting the first letter 3, we get $\mathcal{H} \in \mathcal{S}(\mathcal{H}, y)$, a contradiction with Proposition 6.

There also exist examples where each letter occurs infinitely many times:

Example 10. The following word:

$$x = 012001120001112 \dots 0^k 1^k 2 \dots$$

does not have two words y, z in its shift orbit closure such that $x \in \mathcal{S}(y, z)$. The idea of the proof is that the shift orbit closure consists of words of the following form: $1^* 20^\omega$, $0^* 1^\omega$, x itself and all their right shifts. Shuffling any two words of those types, it is not hard to see that there exists a prefix of the shuffle which contains too many or too few occurrences of some letter compare to the prefix of x . We leave the details of the proof to the reader.

By Corollary 8, there are x, y, z in the shift orbit closure of \mathcal{H} such that $x \in \mathcal{S}(y, z)$. To conclude this section, we give an explicit construction of two words in the shift orbit closure of \mathcal{H} which can be shuffled to give \mathcal{H} . We remark though that this construction gives a shuffle different from the one given by Corollary 8. Let:

$$h : \begin{cases} 0 \mapsto 012 \\ 1 \mapsto 02 \\ 2 \mapsto 1 \end{cases} \quad \text{and} \quad h' : \begin{cases} 0 \mapsto 210 \\ 1 \mapsto 20 \\ 2 \mapsto 1. \end{cases}$$

By definition, the shift orbit closure of the Hall word is closed under h . Moreover this shift orbit closure is also closed under h' , since the factors of the Hall word are closed under the morphism $0 \rightarrow 2, 1 \rightarrow 1, 2 \rightarrow 0$.

$$h' \circ h : \begin{cases} 0 \mapsto 210201 \\ 1 \mapsto 2101 \\ 2 \mapsto 20 \end{cases} \quad h \circ h' : \begin{cases} 0 \mapsto 102012 \\ 1 \mapsto 1012 \\ 2 \mapsto 02 \end{cases} \quad h^2 : \begin{cases} 0 \mapsto 012021 \\ 1 \mapsto 0121 \\ 2 \mapsto 02 \end{cases} \quad h'^2 : \begin{cases} 0 \mapsto 120210 \\ 1 \mapsto 1210 \\ 2 \mapsto 20. \end{cases}$$

Note that if w is an infinite word, then $2(h \circ h')(w) = (h' \circ h)(w)$ and $0h'^2(w) = h^2(w)$.

Theorem 11. $h^\omega(0) \in \mathcal{S}(h^2((h'^2)^\omega(1)), h'^3(h^\omega(0)))$.

Proof. Let

$$U_0 = 01, U_1 = h'(0), U_2 = h'(1), V_0 = h'(1),$$

and for every $i \geq 0$,

$$U_{i+3} = h'^2(h^i(1)) \text{ and } V_{i+1} = h'^2(h^i(1)).$$

Let furthermore

$$u = \prod_{i=0}^{\infty} U_i, v = \prod_{i=0}^{\infty} V_i, \text{ and } w = \prod_{i=0}^{\infty} U_i V_i.$$

We show that $w = h^\omega(0)$, $u = h^2((h'^2)^\omega(1))$ and $v = h'^3(h^\omega(0))$.

Note that $2h(h'(h^\omega(0))) = h'(h^\omega(0))$, thus $h'(h^\omega(0)) = \prod_{i=0}^{\infty} h^i(2)$. Then we have

$$v = 20 \prod_{i=0}^{\infty} h'^2(h^i(1)) = h'^2 \left(\prod_{i=0}^{\infty} h^i(2) \right) = h'^3(h^\omega(0)).$$

Moreover,

$$\begin{aligned} u &= 0121020 \prod_{i=0}^{\infty} h'^2(h^i(1)) = 01210h'^2 \left(\prod_{i=0}^{\infty} h^i(2) \right) \\ &= 01210h'^3(h^\omega(0)) = 01h'(0h'^2(h^\omega(0))) = 01h'(h^\omega(0)) = 0h'(2h^\omega(0)). \end{aligned}$$

Since $h'^2(2h^\omega(0)) = 20h'^2(h^\omega(0)) = 2h^\omega(0)$, the word $2h^\omega(0)$ is the fixed point $(h'^2)^\omega(2)$ of h'^2 , and then $h'(2h^\omega(0))$ is the fixed point $(h'^2)^\omega(1)$. Thus $u = 0(h'^2)^\omega(1) = h^2((h'^2)^\omega(1))$. Finally:

$$w = 0120210121020 \prod_{i=0}^{\infty} h'^2(h^i(021)) = 012021h(021)h^2 \left(\prod_{i=0}^{\infty} h^i(021) \right) = 012 \prod_{i=0}^{\infty} h^i(021).$$

Applying the morphism h to the second expression for w , we get

$$h(w) = 012021h \left(\prod_{i=0}^{\infty} h^i(021) \right) = 012 \prod_{i=0}^{\infty} h^i(021).$$

Thus $w = h^\omega(0)$ since h is injective. □

5 Conclusion and open question

We showed that infinite square-free self-shuffling words exist. The natural question that arises now is whether we can find infinite self-shuffling words subject to even stronger avoidability constraints: For this we recall the notion of *repetition threshold* $RT(k)$, which is defined as the least real number such that an infinite word over Σ_k exists, that does not contain repetitions of exponent greater than $RT(k)$. Due to the collective effort of many researchers (see [3, 8] and references therein), the repetition threshold for all alphabet sizes is known and characterized as follows:

$$RT(k) = \begin{cases} \frac{7}{4} & \text{if } k = 3 \\ \frac{7}{5} & \text{if } k = 4 \\ \frac{k}{k-1} & \text{else.} \end{cases}$$

A word $w \in \Sigma_k^\omega$ without factors of exponent greater than $RT(k)$ is called a *Dejean word*. Charlier et al. showed that the Thue-Morse word, which is a binary Dejean word, is self-shuffling [2].

Question 12. Do there exist self-shuffling Dejean words over non-binary alphabets?

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